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TRANSVERSE VIBRATIONS OF A CIRCULAR ANNULAR PLATE WITH AN INTERMEDIATE CIRCULAR SUPPORT AND A FREE INNER EDGE

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1. Introduction

Continuous, rectangular plates executing transverse vibrations have been studied by several investigators [1, 2]. The case of a solid circular plate with internal circular or secant supports has been treated in reference [3].

The present study deals with the determination of the fundamental frequency of transverse vibration of the structural system shown in Figure 1. Three different approaches are used in order to obtain approximate values of the fundamental frequency coefficients: the optimized Rayleigh–Ritz (R–R) method [4–6]; the finite element method using a standard, well-known code [7]; the differential quadrature (DQ) method [8].

The optimized R–R method is formulated taking into account the existence of an outer edge elastically restrained against rotation (Figure 1) but the numerical results are obtained for the clamped or simply supported limiting cases.

2. Analytical solution by means of the optimized Rayleigh–Ritz method

Following previous studies [6], it is convenient to approximate the fundamental mode shape of the structural element shown in Figure 1 using the expression

$$W(r) \simeq W_a(r) = C_1(\alpha_P r^P + \alpha_Q r^Q + \alpha_2 r^2 + 1) + C_2(\beta_P r^{P+1} + \beta_Q r^{Q+1} + \beta_3 r^3 + 1), \quad (1)$$

where P and Q are Rayleigh's optimization parameters and where the α's and β's are determined substituting each co-ordinate function in the boundary conditions

$$W(a) = W(c) = 0, \qquad \left.\frac{\mathrm{d}W}{\mathrm{d}r}\right|_{r=a} = -\phi D\left.\left(\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{\mu}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right|_{r=a}. \quad \text{(2a, b)}$$

The natural boundary conditions are $r = b$ are not satisfied.

For the present case they are

$$M_r(b) = Q_r(b) = 0, \quad (3)$$

where $M_r(b)$ is the radial bending moment and $Q_r(b)$ is the shear force.

This is certainly a legitimate procedure, since use will be made of the classical Rayleigh–Ritz method. On the other hand, since the condition that the shear force is null, is not used, the present procedure possesses the additional advantage that it is valid for the situation where $b = c$ or, in other words, the plate is simply supported at $r = b = c$.

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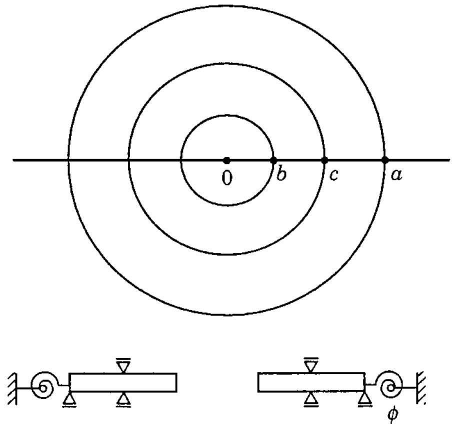

Figure 1. The vibrating system under study: free edge at $r = b$, ϕ = flexibility coefficient.

Substituting equation (1) in the governing functional

$$J[W] = \frac{D}{2}\int_b^a\int_0^{2\pi}\left\{\left(\frac{\mathrm{d}^2W}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 - 2(1-\nu)\left[\left(\frac{\mathrm{d}^2W}{\mathrm{d}r^2}\right)\left(\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right]\right\}r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$-\frac{\rho h}{2}\omega^2\int_b^a\int_0^{2\pi} W^2 r\,\mathrm{d}r\,\mathrm{d}\theta \qquad (4)$$

and requiring that

$$\frac{\partial J[W]}{\partial C_i} = 0 \qquad (i = 1, 2), \qquad (5)$$

one obtains an homogeneous, linear system of equations in the C_i's. The non-triviality condition yields a determinantal equation the lowest root of which constitutes the fundamental frequency coefficient of the structural system shown in Figure 1, $\Omega_1 = \sqrt{\rho h/D}\omega_1 a^2$.

Since the method yields upper bounds, one can minimize Ω_1 with respect to P and Q (either independently keeping constant one of them or with respect to both parameters simultaneously). The value of Ω_1 is then optimized.

3. APPLICATION OF THE DQ METHOD

As it is well known, the classical treatment of transverse vibrations of the system depicted in Figure 1 constitutes satisfying the differential system,

$$D\left(W^{\mathrm{IV}} + 2\frac{W'''}{\bar{r}} - \frac{W''}{\bar{r}^2} + \frac{W'}{\bar{r}^3}\right) - \rho h\omega^2 W = 0, \qquad (6)$$

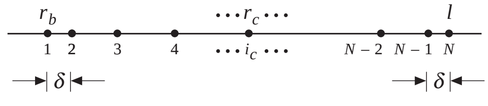

Figure 2. Partition of the domain when using the differential quadrature technique.

TABLE 1

Values of Ω_1 in the case in which the inner boundary is simply supported ($b = c$): comparison of results

Boundary condition	Method	b/a				
		0·1	0·2	0·3	0·4	0·5
Simply	[10]*	14·5	—	21·1	—	40·00
supported	FE	14·44	16·73	21·04	28·09	40·02
outer	[9]†	14·44	17·39	21·31	28·25	40·01
boundary	R–R	14·52	16·80	21·08	28·15	40·33
Clamped	[10]*	22·6	—	33·7	—	63·9
outer	FE	22·58	26·62	33·66	44·94	63·87
boundary	[9]†	22·61	26·57	33·66	44·89	64·06
	R–R	22·93	26·74	33·79	45·21	64·90

* References [10] states that the eigenfrequency is independent of the Poisson ratio. This does not appear to be a correct statement.
† Determined for $v = 1/3$.

$$W''(b) + \frac{v}{b} W'(b) = 0, \qquad W'''(b) + \frac{W''(b)}{b} - \frac{W'(b)}{b^2} = 0, \qquad W(c) = 0; \tag{7a–c}$$

$$W(a) = 0, \qquad W'(a) = 0, \qquad \text{(clamped outer edge)}; \tag{7d, e}$$

$$W(a) = 0, \qquad W''(a) + \frac{v}{a} W'(a) = 0 \qquad \text{(simply supported outer edge)}. \tag{8a, b}$$

Introducing the dimensionless variable $r = \bar{r}/a$ and substituting in equation (6), one obtains

$$W^{\mathrm{IV}} + 2\frac{W'''}{r} - \frac{W''}{r^2} + \frac{W'}{r^3} - \Omega^2 W = 0 \quad \left(\Omega^2 = \frac{\rho h a^4 \omega^2}{D}\right). \tag{9}$$

The boundary conditions become

$$W'''(r_b) + \frac{v}{r_b} W'(r_b) = 0, \qquad W'''(r_b) + \frac{W''(r_b)}{r_b} - \frac{W'(r_b)}{r_b^2} = 0; \tag{10a, b}$$

$$W(r_c) = 0, \qquad r_b = b/a, \qquad r_c = c/a; \tag{10c}$$

$$W(1) = 0, \qquad W'(1) = 0, \qquad \text{(clamped outer edge)}; \tag{10d, e}$$

$$W(1) = 0, \qquad W''(1) + vW'(1) = 0, \qquad \text{(simply supported outer edge)}. \tag{11a, b}$$

Using Bert's well-established notation [8] one denotes by A_{ik}, B_{ik}, C_{ik} and D_{ik} the coefficients of the linear combinations of first, second, third and fourth order derivatives, respectively. Application of the DQ method leads, then, to the following linear system of homogeneous equations in the discrete values of the displacement amplitude, W_i (see Figure 2):

$$\sum_{k=1}^{N=1}\left(D_{ik}+\frac{2}{r_i}C_{ik}-\frac{1}{r_i^2}B_{ik}+\frac{1}{r_i^3}A_{ik}\right)W_k-\Omega^2 W_i=0, \qquad i=3,\ldots,N-2, \quad i\neq i_c, \tag{12a}$$

$$\sum_{k=1}^{N-1}\left(B_{ik}+\frac{\nu}{r_1}A_{1k}\right)W_k=0, \tag{12b}$$

$$\sum_{k=1}^{N-1}\left(C_{2k}+\frac{1}{r_2}B_{2k}-\frac{1}{r_2^2}A_{2k}\right)W_k=0, \tag{12c}$$

$$\sum_{k=1}^{N-1}\left(B_{(N-1)k}+\frac{\nu}{r_{N-1}}A_{(N-1)k}\right)W_k=0, \tag{12d}$$

when the plate is simply supported at the outer edge. A similar system of linear, homogeneous equations is obtained when the plate is clamped at the outer edge. Following reference [8] the calculations performed in the present study have been performed making $\delta = 10^{-4}$ (Figure 2).

4. NUMERICAL RESULTS

All calculations have been performed for ν (the Poisson ratio) = 0·3. The frequency determinations carried out by means of the DQ method have been concerned with the configurations $b/a = 0{\cdot}2$ and 0·4 of Figure 1 ($N = 10$ and 15 respectively). For the finite

TABLE 2

Values of Ω_1 in the case of an annular plate of dimensions b/a as the concentric inner support varies its position (c/a): comparison of results ($\nu = 0{\cdot}3$)

Boundary condition	b/a	Method	c/a 0·3	0·4	0·5	0·6
Simply supported outer edge	0·2	DQ	20·06	25·42	34·85	—
		FE	22·03	28·52	30·70	23·87
		R–R	22·17	28·96	30·93	24·42
	0·4	DQ	—	—	35·78	52·80
		FE	—	—	39·97	46·39
		R–R	—	—	40·28	46·56
Clamped outer edge	0·2	DQ	31·78	40·81	—	—
		FE	34·70	42·63	35·92	25·50
		R–R	34·91	43·01	36·47	26·20
	0·4	DQ	—	—	57·53	—
		FE	—	—	62·60	56·63
		R–R	—	—	63·32	57·64

TABLE 3

Fundamental frequency coefficients Ω_1 in the case of a solid circular plate with a concentric circular support at $r = c$: comparison of results ($\nu = 0{\cdot}30$)

	Clamped		Simply supported	
c/a	Reference [3]	DQ	Reference [3]	DQ
0·1	25·03	24·13	16·22	15·67
0·2	30·02	27·31	19·34	17·62
0·3	36·99	32·69	23·96	20·84
0·4	39·60	42·35	28·95	26·28

element determinations 6400 elements were considered (some cases were tested with an 8000-element modelling and no significant differences were noticed).

Table 1 depicts a comparison of fundamental eigenvalues for the case where $b/c = 1$ or, in other words, where the inner boundary is simply supported. The agreement between the eigenfrequencies determined by the R–R and FE approaches is quite satisfactory. In general the R–R method yields results which are, at most, 2% higher than the values obtained by means of the finite element technique.

Table 2 shows a comparison of values of Ω_1 for $b/a = 0{\cdot}2$ and 0·4. For the first case the parameter c/a was taken equal to 0·3, 0·4, 0·5 and 0·6, while for the second $c/a = 0{\cdot}5$ and 0·6.

It is observed that for $c/a = 0{\cdot}5$ the DQ results differ by approximately 10% from the values obtained by means of the R–R and finite element methods which, on the other hand, are in very good agreement from an engineering viewpoint.

For $c/a = 0{\cdot}6$ the eigenfrequencies determined using the DQ technique were not satisfactory.

The DQ method has also been applied in the present study to the case of solid circular plates of radius a and a concentric circular support at $r = c$.

A comparison with the results obtained in reference [3] is presented in Table 3. The maximum differences with the values available in the literature are of the order of 10% and occur for $c/a = 0{\cdot}3$.

For $c/a > 0{\cdot}4$ the DQ technique did not provide satisfactory accuracy.

Tables 4 and 5 depict fundamental frequency coefficients for simply supported and clamped outer boundaries, respectively. The tables allow for the evaluation of the following:

(1) The convergence of the procedure as one and two polynomial co-ordinate functions are used, column (A).

(2) The effect of optimizing Ω_1 with respect to the exponential parameter P, as Q is kept constant, column (B), when two co-ordinate functions are used.

(3) The effect of optimizing Ω_1 with respect to Q, as P is kept constant, when two co-ordinate functions are used.

It is observed that in some instances (2) is more effective than (3); in others the situation reverses, and in some cases they are equivalent.

It is interesting to notice the fact that as c/a approaches unity the value of Ω_1 is practically the same, regardless the type of boundary condition at the outer boundary (see, for instance, the case in which $b/a = 0{\cdot}5$, $c/a = 0{\cdot}9$: $\Omega_1 = 23{\cdot}48$, Table 4, and $\Omega_1 = 24{\cdot}23$, Table 5).

TABLE 4

Circular anular plate with an outer simply supported boundary

		(A)				(B)			(C)		
b/a	c/a	P	Q	$\Omega^{(1)}$	$\Omega^{(2)}$	P	Q	$\Omega^{(2)}$	P	Q	$\Omega^{(2)}$
0·1	0·1	4	3	15·95	15·14	0·1	3	14·51	4	1·2	14·53
	0·2			18·67	18·48	0·8		18·44		2·7	18·47
	0·3			23·56	23·50	1·7		23·44		1·2	23·46
	0·4			29·21	28·99	2·2		28·95		1·4	28·88
	0·5			28·56	28·42	1·1		28·19		0·7	28·22
	0·6			22·65	22·61	1·2		22·47		7·3	22·42
	0·7			17·94	17·78	2·1		17·75		1·4	17·68
	0·8			15·16	14·42	7·1		14·37		9·3	14·37
	0·9			13·71	12·45	15·4		11·99		17·2	12·01
0·2	0·2	4	3	17·58	16·91	2·3	3	16·80	4	1·9	16·79
	0·3			22·24	22·20	8·7		22·17		3·9	22·20
	0·4			29·13	29·01	2·3		28·99		1·5	28·96
	0·5			31·22	31·06	1·1		30·93		0·7	30·94
	0·6			24·75	24·49	8·9		24·42		2·7	24·48
	0·7			18·92	18·91	1·6		18·83		1·1	18·76
	0·8			15·55	15·23	10·2		15·12		14·9	15·17
	0·9			13·85	13·11	19·7		12·52		21·9	12·60
0·3	0·3	4	3	21·26	21·08	5·2	3	21·08	4	3·5	21·07
	0·4			29·10	28·93	1·3		28·88		0·7	28·91
	0·5			37·18	36·90	3·2		36·89		12·7	36·83
	0·6			31·60	31·53	9·7		31·38		6·1	31·34
	0·7			22·84	22·80	10·1		22·76		12·7	22·64
	0·8			17·87	17·41	9·6		17·33		15·2	17·35
	0·9			15·46	14·54	20·6		13·91		23·3	13·99
0·4	0·4	4	3	28·31	28·27	0·1	3	28·17	4	8·2	28·15
	0·5			42·37	40·39	1·9		40·28		1·2	40·31
	0·6			47·44	46·70	1·1		46·56		10·6	46·56
	0·7			33·13	32·65	0·6		32·48		0·1	32·53
	0·8			24·01	22·60	5·6		22·58		8·5	22·58
	0·9			19·87	17·72	17·9		17·05		19·7	17·08
0·5	0·5	4	3	41·36	40·38	0·1	3	40·10	4	0·1	40·17
	0·6			68·28	60·78	2·9		60·73		2·1	60·74
	0·7			59·00	56·34	0·1		55·71		0·1	55·84
	0·8			38·18	34·25	3·3		34·24		2·1	34·23
	0·9			29·48	24·22	15·1		23·48		15·3	23·48
0·6	0·6	4	3	66·85	62·65	0·1	3	62·22	4	0·1	62·30
	0·7			117·8	99·70	3·6		99·70		2·7	99·70
	0·8			74·18	64·52	0·8		64·13		0·1	64·16
	0·9			50·39	38·17	12·8		37·37		11·9	37·36
0·7	0·7	4	3	123·8	110·7	0·1	3	110·1	4	0·1	110·2
	0·8			198·5	167·0	0·1		166·4		0·1	166·4
	0·9			103·7	75·07	9·6		74·50		8·1	74·48
0·8	0·8	4	3	290·8	247·8	0·2	3	247·2	4	0·2	247·3
	0·9			319·2	240·2	1·8		239·4		1·8	239·6
0·9	0·9	4	3	1210	988·1	0·1	3	987·1	4	0·1	987·2

Notes $\Omega^{(1)}$ determined with one co-ordinate function; $\Omega^{(2)}$ determined with two co-ordinate functions. (A) results obtained without optimization; (B) results obtained optimizing with respect to P; (C) results obtained optimizing with respect to Q.

TABLE 5

Circular anular plate with an outer clamped boundary

		(A)				(B)			(C)		
b/a	c/a	P	Q	$\Omega^{(1)}$	$\Omega^{(2)}$	P	Q	$\Omega^{(2)}$	P	Q	$\Omega^{(2)}$
0·1	0·1	4	3	24·96	23·44	0·1	3	22·88	4	1·5	22·78
	0·2			29·23	28·82	5·7		28·77		3·7	28·78
	0·3			36·37	36·36	8·7		36·13		6·9	35·92
	0·4			40·18	40·04	1·1		39·76		0·7	39·84
	0·5			32·56	31·91	6·3		31·76		4·2	31·81
	0·6			24·10	24·10	9·7		23·88		7·7	23·74
	0·7			19·00	18·58	3·3		18·57		12·4	18·43
	0·8			16·08	14·97	7·9		14·81		9·2	14·82
	0·9			14·34	12·89	18·2		12·20		19·4	12·22
0·2	0·2	4	3	27·86	26·79	3·2	3	26·75	4	2·5	26·75
	0·3			35·18	35·12	8·2		34·93		5·3	34·91
	0·4			43·18	43·18	9·8		43·01		6·1	43·12
	0·5			37·47	36·52	5·2		36·47		3·4	36·49
	0·6			26·55	26·40	9·1		26·20		5·4	26·28
	0·7			20·01	19·89	2·3		19·86		1·4	19·78
	0·8			16·48	15·88	10·4		15·65		14·1	15·71
	0·9			14·49	13·60	21·7		12·75		23·3	12·83
0·3	0·3	4	3	34·13	33·91	6·6	3	33·79	4	4·7	33·77
	0·4			46·50	46·04	0·9		45·75		9·1	45·49
	0·5			50·85	50·85	8·8		50·63		6·1	50·51
	0·6			35·39	35·37	9·8		34·98		7·2	34·74
	0·7			24·58	24·28	2·5		24·26		12·7	24·04
	0·8			19·17	18·24	9·7		18·05		13·8	18·08
	0·9			16·32	15·14	22·1		14·21		24·1	14·29
0·4	0·4	4	3	45·79	45·67	0·1	3	45·29	4	8·6	45·21
	0·5			67·72	63·80	1·3		63·32		0·5	63·41
	0·6			59·10	58·54	0·1		57·64		9·8	57·69
	0·7			36·92	35·57	1·1		35·29		12·4	35·29
	0·8			26·39	23·84	6·7		23·74		7·2	23·74
	0·9			21·33	18·56	19·9		17·49		21·2	17·52
0·5	0·5	4	3	67·09	65·10	0·1	3	64·28	4	0·1	64·55
	0·6			105·9	93·69	1·7		93·22		0·8	93·28
	0·7			69·43	65·06	0·1		63·67		0·1	64·01
	0·8			43·14	36·51	4·2		36·50		3·4	36·50
	0·9			32·31	25·54	17·4		24·24		17·4	24·23
0·6	0·6	4	3	108·4	100·5	0·1	3	99·32	4	0·1	99·62
	0·7			166·9	143·4	0·6		142·4		0·1	142·5
	0·8			86·09	70·47	1·4		70·10		0·8	70·16
	0·9			56·42	40·55	15·3		38·95		14·5	38·92
0·7	0·7	4	3	201·0	176·5	0·1	3	175·0	4	0·1	175·3
	0·8			247·9	206·5	0·1		203·6		0·1	204·2
	0·9			119·0	80·41	12·1		78·89		11·3	78·82
0·8	0·8	4	3	472·2	392·8	0·2	3	390·7	4	0·2	391·2
	0·9			377·0	263·2	2·9		261·9		1·8	261·9
0·9	0·9	4	3	1970	1544	0·1	3	1546	4	0·4	1548

Notes $\Omega^{(1)}$ determined with one co-ordinate function; $\Omega^{(2)}$ determined with two co-ordinate functions. (A) results obtained without optimization; (B) results obtained optimizing with respect to P; (C) resubts obtained optimizing with respect to Q.

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